The Polynomial Form of the Scattering Equations

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Abstract

The scattering equations, recently proposed by Cachazo, He and Yuan as providing a kinematic basis for describing tree amplitudes for massless particles in arbitrary space-time dimension (including scalars, gauge bosons and gravitons), are reformulated in polynomial form. The scattering equations for N particles are shown to be equivalent to a Möbius invariant system of $N-3$ equations, $\tilde{h}_m = 0$, $2 \leq m \leq N-2$, in N variables, where $\tilde{h}_m$ is a homogeneous polynomial of degree m, with the exceptional property of being linear in each variable taken separately. Fixing the Möbius invariance appropriately, yields polynomial equations $h_m = 0$, $1 \leq m \leq N-3$, in $N-3$ variables, where h_m has degree m. The linearity of the equations in the individual variables facilitates computation, *e.g.* the elimination of variables to obtain single variable equations determining the solutions. Expressions are given for the tree amplitudes in terms of the $\tilde{h}_m$ and h_m. The extension to the massive case for scalar particles is described and the special case of four dimensional space-time is discussed.

1 Introduction

In this paper, we study the scattering equations recently proposed by Cachazo, He and Yuan (CHY) [1] as a basis for describing the kinematics of massless particles,

$$f_a(z,k) = 0, \quad a \in A, \qquad \text{where} \quad f_a(z,k) = \sum_{\substack{b\in A\\ b\neq a}} \frac{k_a \cdot k_b}{z_a - z_b}, \tag{1.1}$$

and where k_a are the momenta of the particles labeled by $a \in A$, with $N = |A|$, the number of elements in A, and $k_a^2 = 0$. Typically, we shall take $A = \{1, 2, \ldots, N\}$.

Our principal result is that these equations are equivalent to the homogeneous polynomial equations

$$\sum_{\substack{S\subset A\\ |S|=m}} k_S^2\, z_S = 0, \qquad 2 \le m \le N-2, \tag{1.2}$$

where the sum is over all $N!/m!(N-m)!$ subsets $S \subset A$ with m elements, and

$$k_S = \sum_{b\in S} k_b, \qquad z_S = \prod_{a\in S} z_a, \qquad S \subset A. \tag{1.3}$$

Note that the coefficients, $k_S^2, S \subset A$, in (1.2) are precisely all the Mandelstam variables, *i.e.* the denominators of propagators of tree diagrams for processes involving the particles with momenta $k_a, a \in A$.

We first recall basic properties of the equations (1.1) [1]–[4]. The equations (1.1) are invariant under Möbius transformations,

$$z_a \mapsto \zeta_a = \frac{\alpha z_a + \beta}{\gamma z_a + \delta}, \quad a \in A, \tag{1.4}$$

in that, if $z = (z_a)$ is a solution to (1.1), $\zeta = (\zeta_a)$ also provides a solution, $f_a(\zeta, k) = 0, a \in A$, provided that momentum is conserved and the particles are massless, for, using momentum conservation,

$$\sum_{\substack{b\in A\\ b\neq a}} \frac{k_a \cdot k_b}{\zeta_a - \zeta_b} = \sum_{\substack{b\in A\\ b\neq a}} \frac{(cz_a+d)(cz_b+d)k_a\cdot k_b}{(ad-bc)(z_a-z_b)} = \frac{(cz_a+d)^2}{(ad-bc)} f_a(z,k) + \frac{c(cz_a+d)}{(ad-bc)} k_a^2, \tag{1.5}$$

which vanishes if $f_a(z,k) = 0$ and $k_a^2 = 0$.

The Möbius invariance implies that only $N-3$ of the equations (1.1) are independent. Relations between them are conveniently derived by noting that, if

$$U(z,k) = \prod_{a<b} (z_a - z_b)^{-k_a\cdot k_b}, \quad \text{then} \quad \frac{\partial U}{\partial z_a} = -f_a U. \tag{1.6}$$

and the Möbius invariance of the system of equations (1.1) can be deduced from that of

$U(z,k)$ [4]. Then, considering an infinitesimal Möbius transformation, $\delta z_a = \epsilon_1 + \epsilon_2 z_a + \epsilon_3 z_a^2$, $\delta U = 0$ implies the identities:

$$\sum_{a\in A} f_a(z,k) = 0; \qquad \sum_{a\in A} z_a f_a(z,k) = 0; \qquad \sum_{a\in A} z_a^2 f_a(z,k) = 0. \tag{1.7}$$

So only $N-3$ of the conditions (1.1) are needed to restrict to its set of solutions. This can be achieved by the delta function

$$\prod_{a\in A}{}' \delta\left(f_a(z,k)\right) \equiv (z_i - z_j)(z_j - z_k)(z_k - z_i) \prod_{\substack{a\in A\\ a\neq i,j,k}} \delta\left(f_a(z,k)\right) \tag{1.8}$$

which is independent of the choice of $i,j,k \in A$. Under the Möbius transformation (1.4),

$$\prod_{a\in A}{}' \delta\left(f_a(z,k)\right) \mapsto \prod_{a\in A}{}' \delta\left(f_a(\zeta,k)\right) = \prod_{a\in A} \frac{\alpha\delta - \beta\gamma}{(\gamma z_a + \delta)^2} \prod_{a\in A}{}' \delta\left(f_a(z,k)\right), \tag{1.9}$$

so that the integrand of

$$\mathcal{A}_N = \int \Psi_N(z,k) \prod_{a\in A}{}' \delta\left(f_a(z,k)\right) \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} \Big/ d\omega, \tag{1.10}$$

where the invariant measure on the Möbius group,

$$d\omega = \frac{dz_r dz_s dz_t}{(z_r - z_s)(z_s - z_t)(z_t - z_r)}, \tag{1.11}$$

is Möbius invariant provided that the function $\Psi_N(z,k)$, which may also depend on polarizations, is itself Möbius invariant. [Note that, in (1.10), a cyclic ordering on A has been chosen, with $a+1$ following a, say $(1,2,\ldots,N)$ with $N+1 \equiv 1$.] In order to give a precise interpretation of (1.10) that includes a sum over all, possibly complex, solutions of the equations (1.1), CHY rewrote the expression as a contour integral

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k) \prod_{a\in A}{}' \frac{1}{f_a(z,k)} \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} \Big/ d\omega \tag{1.12}$$

where the contour $\mathcal{O}$ encloses all the solutions of (1.1) and $\prod'$ is defined as in (1.9).

CHY initially proposed a form of $\Psi_N(z,k)$ in (1.10) for tree amplitudes in pure Yang-Mills theory and in gravity in arbitrary space-time dimension [2], subsequently noting that taking Ψ_N constant would yield massless scalar ϕ^3 theory [3]. This result was proved for massless ϕ^3 and pure Yang-Mills theory in [4], where it was noted that the scattering equations (1.1) could be extended to the massive case, $k_a^2 = \mu^2$, and that (1.12) then produced the tree amplitudes of massive ϕ^3 for Ψ_N constant.

It seems that the scattering equations (1.1) were first written down by Fairlie and Roberts [5], who, before the string-theory interpretation of dual resonance models had been established in detail, were seeking a variation of the Veneziano model that was free of tachyons. They subsequently occurred in the work of Gross and Mende on the high energy behavior of string theory [6]. Fairlie and Roberts replaced the Fubini-Veneziano position field, $X^\mu(z)$, and the corresponding momentum field, $P^\mu(z)$,

$$X^\mu(z) = x^\mu - ip^\mu \log z + i\sum_{n\neq 0} \frac{a_n^\mu}{n} z^{-n}, \qquad P^\mu(z) = i\frac{dX^\mu(z)}{dz} = \sum_n a_n^\mu z^{-n-1}, \quad a_0^\mu = p^\mu, \tag{1.13}$$

expanded in terms of an infinite number of harmonic modes, a_n^μ, with

$$x^\mu(z) = -i\sum_{a\in A} k_a^\mu \log(z - z_a), \qquad p^\mu(z) = i\frac{dx^\mu(z)}{dz} = \sum_{a\in A} \frac{k_a^\mu}{z - z_a}. \tag{1.14}$$

The singularities of $x^\mu(z), p^\mu(z)$ on the Riemann sphere are evidently only at $z = z_a$, provided that we have momentum conservation, $\sum_a k_a = 0$. [If not, there will be a singularity at $z = \zeta^{-1} = \infty$, $x \sim -ik_\infty \log\zeta$, where $k_\infty = -\sum_a k_a$.] Fairlie and Roberts replaced the Virasoro conditions, which can be written $\langle : P(z)^2 : \rangle =$ constant, with

$$p(z)^2 = \sum_{a,b} \frac{k_a \cdot k_b}{(z - z_a)(z - z_b)} = 0. \tag{1.15}$$

Given that $p(z)^2$ vanishes sufficiently fast at infinity, it vanishes everywhere provided it has no singularities at $z = z_a$, for $a \in A$. The absence of double poles requires $k_a^2 = 0, a \in A$, and the absence of simple poles implies (1.1). In an approach anticipating in general terms that of [2, 3], they sought to define their model as a sum over the solutions of the scattering equations (1.1), rather than as an integral as in the Veneziano model.

An important property of the scattering equations (1.1) is the way they factorize when one of the Mandelstam variables, k_S^2, vanishes. If $k_S^2 \neq 0$ for all subsets S with between 2 and $N-2$ elements, the values of the $z_a, a \in A$, are distinct. To demonstrate this, suppose that, as the k_a vary (maintaining momentum conservation and the massless conditions), two or more of the z_a to tend to the same value, z_0. Suppose $z_a = z_0 + \epsilon x_a + \mathcal{O}(\epsilon^2)$ for $a \in S$, and $z_a \not\to z_0$ for $a \notin S$, as $\epsilon \to 0$; then the scattering equations factorize into two sets:

$$f_a(z,k) = \frac{1}{\epsilon} f_a^S(x,k)\left[1 + \mathcal{O}(\epsilon)\right], \quad f_a^S(x,k) = \sum_{\substack{b\in S\\ b\neq a}} \frac{k_a \cdot k_b}{x_a - x_b}, \quad a \in S, \tag{1.16}$$

$$f_a(z,k) = \sum_{b\in S} \frac{k_a \cdot k_b}{z_a - z_0} + \sum_{\substack{b\notin S\\ b\neq a}} \frac{k_a \cdot k_b}{z_a - z_b} + \mathcal{O}(\epsilon), \quad a \notin S, \tag{1.17}$$

implying in the limit that $f_a^S(x,k)=0, a\in S$, and

$$k_S^2 = 2\sum_{\substack{a,b\in S\\ b\neq a}} \frac{x_a}{x_a-x_b} k_a\cdot k_b = 2\sum_{a\in S} x_a \sum_{\substack{b\in S\\ b\neq a}} \frac{k_a\cdot k_b}{x_a-x_b} = 2\sum_{a\in S} x_a f_a^S(x,k) = 0. \tag{1.18}$$

In what follows, in general, we shall assume $k_S^2\neq 0$ for all subsets S with between 2 and $N-2$ elements and, consequently, that $z_a\neq z_b$ for $a\neq b$, $a,b\in A$.

In section 2, we derive the polynomial form (1.2) of the scattering equations, first directly and then, more elegantly, by using their equivalence to (1.15). We then show show that the polynomial form possesses a factorization property corresponding to (1.16) and (1.17). For the special kinematic configuration in which k_S^2 is independent of $S\subset A$ if $|S|=m$, the solutions of (1.2) are given by complex roots of unity, which we describe in section 3, where we also show how the fact that (1.2) is linear in each z_a taken separately enables simple forms for single-variable equations to be found which determine the solutions to the scattering equations, at least for $N\leq 6$. Expressions for the scattering amplitudes in terms of the polynomials (1.2) are given in section 4. The Möbius invariance of the integrand is demonstrated in section 5, where it shown how the requirement of Möbius invariance on equations of the form

$$\sum_{\substack{S\subset A\\ |S|=m}} \lambda_S\, z_S = 0, \qquad 2\leq m\leq N-2, \tag{1.19}$$

implies that $\lambda_S = k_S^2$, for suitable light-like momenta $k_a, a\in A$. The extension to the massive case is given in section 6, and special features of the four-dimensional space are discussed in section 7. Some comments and indications of further directions for investigation are given in section 8.

2 Polynomial Form for the Scattering Equations

The Polynomial Equations. Studies of the scattering equations indicate that, after using Möbius invariance to fix three of the z_a, they have $(N-3)!$ solutions [1, 2]. Bézout's Theorem states that number of solutions to a system of polynomial equations, possessing a finite number of solutions, is bounded by the product of their degrees and, suitably counted, the number will attain this bound (see, *e.g.* [7]). This suggests that the scattering equations (1.1) should be equivalent to $N-3$ homogeneous equations in the z_a, $h_m=0,\ 1\leq m\leq N-3$, where h_m has degree m.

Writing

$$g_m(z,k) = \sum_{a\in A} z_a^{m+1} f_a(z,k), \tag{2.1}$$

in (1.7), we noted that Möbius invariance implies that g_{-1}, g_0 and g_1 vanish identically. For other integral values of m, (2.1) defines an homogeneous polynomial of degree m in the z_a,

$$g_m(z,k) = \sum_{\substack{a,b\in A\\ a\neq b}} \frac{k_a\cdot k_b z_a^{m+1}}{z_a - z_b} = \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b \frac{z_a^{m+1} - z_b^{m+1}}{z_a - z_b}$$
$$= \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b \sum_{r=0}^{m} z_a^r z_b^{m-r}. \tag{2.2}$$

The $N\times N$ matrix $Z_{ma} = z_a^{m+1}, a\in A, -1\leq m\leq N-2$, is nonsingular, given that $z_a\neq z_b$ for $a\neq b$, because $\det Z$ is the Vandermonde determinant,

$$\det Z = \prod_{1\leq a<b\leq N} (z_a - z_b). \tag{2.3}$$

Thus the $N-3$ equations

$$g_m = 0, \qquad \text{where} \quad 2\leq m\leq N-2, \tag{2.4}$$

are equivalent to the equations $f_a = 0, a\in A$, $N-3$ of which are independent. Since $g_m = 0$, for all integers m, follows from $f_a = 0, a\in A$, and these equations themselves follow from the set (2.4), it follows that this set implies $g_n = 0$ for all integers n.

The set of $N-3$ homogeneous polynomial equations (2.4) is equivalent to the scattering equations (1.1) but this form is not convenient, *e.g.* for taking advantage of the Möbius invariance to fix z_1 at ∞. We can find a form of these polynomials that is more convenient as follows. Write $k_{a_1a_2\ldots a_n} = k_{a_1} + k_{a_2} + \cdots + k_{a_n}$,

$$g_2(z,k) = \sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b\, z_a^2 + \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b\, z_a z_b$$
$$= -\sum_{a\in A} k_a^2 z_a^2 + \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b z_a z_b = \frac{1}{4}\sum_{\substack{a,b\in A\\ a\neq b}} k_{ab}^2 z_a z_b. \tag{2.5}$$

$$g_3(z,k) = \sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b z_a^3 + \sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b z_a^2 z_b$$
$$= -\sum_{a\in A} k_a^2 z_a^3 + \sum_{\substack{a,b\in A\\ a\neq b}} k_a\cdot k_b z_a^2 z_b = \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_{ab}^2 z_a^2 z_b. \tag{2.6}$$

$$g_2(z,k)\sum_{c\in A} z_c = \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_{ab}^2 z_a^2 z_b + \frac{1}{4}\sum_{\substack{a,b,c\in A\\ a,b,c\ \text{uneq.}}} k_{ab}^2 z_a z_b z_c$$

$$= \frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}} k_{ab}^2 z_a^2 z_b + \frac{1}{12}\sum_{\substack{a,b,c\in A\\ a,b,c\ \text{uneq.}}} k_{abc}^2 z_a z_b z_c, \tag{2.7}$$

since $k_{abc}^2 = k_{ab}^2 + k_{bc}^2 + k_{ac}^2$, given that $k_b^2 = 0$ for $b \in A$. Thus,

$$\sum_{\substack{c,a\in A\\ c\neq a}} z_c z_a^3 f_a(z,k) = g_2(z,k)\sum_{c\in A} z_c - g_3(z,k) = \frac{1}{12}\sum_{\substack{a,b,c\in A\\ a,b,c\ \text{uneq.}}} k_{abc}^2 z_a z_b z_c. \tag{2.8}$$

More generally,

$$\begin{aligned}
&\sum_{\substack{a_0,a_2,\ldots,a_m\in A\\ a_i\ \text{uneq.}}} z_{a_2}\ldots z_{a_m} z_{a_0}^2 f_{a_0}\\
&\quad= \sum_{\substack{a_0,a_1,\ldots,a_m\in A\\ a_i\ \text{uneq.}}} \frac{k_{a_0}\cdot k_{a_1} z_{a_0}^2 z_{a_2}\ldots z_{a_m}}{z_{a_0}-z_{a_1}} + (m-1)\sum_{\substack{a_0,a_1,\ldots,a_{m-1}\in A\\ a_i\ \text{uneq.}}} \frac{k_{a_0}\cdot k_{a_1} z_{a_0}^2 z_{a_1}\ldots z_{a_{m-1}}}{z_{a_0}-z_{a_1}}\\
&\quad= \sum_{\substack{a_0,a_1,\ldots,a_m\in A\\ a_i\ \text{uneq.}}} k_{a_0}\cdot k_{a_1} z_{a_0} z_{a_2}\ldots z_{a_m} + \frac{m-1}{2}\sum_{\substack{a_0,a_1,\ldots,a_{m-1}\in A\\ a_i\ \text{uneq.}}} k_{a_0}\cdot k_{a_1} z_{a_0} z_{a_1}\ldots z_{a_{m-1}}\\
&\quad= -\frac{m-1}{2}\sum_{\substack{a_0,a_1,\ldots,a_{m-1}\in A\\ a_i\ \text{uneq.}}} k_{a_0}\cdot k_{a_1} z_{a_0} z_{a_1}\ldots z_{a_{m-1}}.
\end{aligned} \tag{2.9}$$

So

$$\begin{aligned}
\sum_{\substack{a_0,a_2,\ldots,a_m\in A\\ a_i\ \text{uneq.}}} z_{a_2}\ldots z_{a_m} z_{a_0}^2 f_{a_0} &= -\frac{1}{m}\sum_{\substack{a_1,a_2,\ldots,a_m\in A\\ a_i\ \text{uneq.}}}\sum_{\substack{i,j=1\\ i<j}}^{m} k_{a_i}\cdot k_{a_j} z_{a_1} z_{a_2}\ldots z_{a_m}\\
&= -\frac{1}{2m}\sum_{\substack{a_1,a_2,\ldots,a_m\in A\\ a_i\ \text{uneq.}}} k_{a_1a_2\ldots a_m}^2 z_{a_1} z_{a_2}\ldots z_{a_m},\\
&\qquad\qquad \text{which vanishes unless } 2\le m\le N-2,\\
&= -\frac{(m-1)!}{2}\sum_{\substack{S\subset A\\ |S|=m}} k_S^2 z_S,
\end{aligned} \tag{2.10}$$

where $|S|$ denotes the number of elements of S, and k_S, z_S are defined by (1.3).

The homogeneous polynomials

$$\tilde h_m = \sum_{\substack{S\subset A\\ |S|=m}} k_S^2 z_S = \frac{1}{m!}\sum_{\substack{a_1,a_2,\ldots,a_m\in A\\ a_i\ \text{uneq.}}} k_{a_1a_2\ldots a_m}^2 z_{a_1} z_{a_2}\ldots z_{a_m}, \quad 2\le m\le N-2, \tag{2.11}$$

are quite simply related to the g_m defined by (2.1) because

$$\sum_{\substack{a_0,a_2,\dots,a_m\in A\\ a_i\ \text{uneq.}}} z_{a_2}\dots z_{a_m} z_{a_0}^2 f_{a_0} = \sum_{\substack{a_2,\dots,a_m\in A\\ a_i\ \text{uneq.}}} z_{a_2}\dots z_{a_m}\sum_{a_0\in A} z_{a_0}^2 f_{a_0} - (m-1)\sum_{\substack{a_0,a_3,\dots,a_m\in A\\ a_i\ \text{uneq.}}} z_{a_3}\dots z_{a_m} z_{a_0}^3 f_{a_0}$$
$$= \sum_{r=2}^{m+1}\frac{(-1)^r(m-1)!}{(m-r+1)!}\sum_{\substack{a_r,\dots,a_m\in A\\ a_i\ \text{uneq.}}} z_{a_r}\dots z_{a_m}\sum_{a_0\in A} z_{a_0}^r f_{a_0}, \tag{2.12}$$

so that, since g_1 vanishes identically,

$$\tilde{h}_m = 2\sum_{r=2}^{m}(-1)^r g_r \Sigma^A_{m-r}, \quad 2\le m\le N-2, \tag{2.13}$$

where Σ^A_r is the elementary symmetric function

$$\Sigma^A_r = \sum_{\substack{S\subset A\\ |S|=r}} z_S = \frac{1}{r!}\sum_{\substack{a_1,\dots,a_r\in A\\ a_i\ \text{uneq.}}} z_{a_1}\dots z_{a_r}. \tag{2.14}$$

Since the equations $f_a = 0, a\in A$, are equivalent to the equations $g_m = 0,\ 2\le m\le N-2$, they are also equivalent to the equations, $\tilde{h}_m = 0,\ 2\le m\le N-2$. $\tilde{h}_m$ is a homogeneous polynomial of degree m with the special property that, although it is of degree m in the z_a, it is linear in each one of them taken individually, *i.e.* the monomials that it comprises are square-free. This considerably simplifies various calculations and constructions.

The coefficients, $k^2_{a_1a_2\dots a_m}$, in the $\tilde{h}_m$ are all determined in terms of the $\frac{1}{2}N(N-1)$ coefficients $k^2_{ab} = 2k_a\cdot k_b$,

$$k^2_U = \sum_{\substack{S\subset U\\ |S|=2}} k^2_S. \tag{2.15}$$

Of course, these are not independent but must satisfy the constraints of momentum conservation. This leaves $\frac{1}{2}N(N-3)$ independent $k_a\cdot k_b$, provided that $N\le D+1$, so that the constraints of the dimension D of space-time do not enter. If $N > D+1$, the number of independent $k_a\cdot k_b$ is reduced to $N(D-1)-\frac{1}{2}D(D+1)$.

An Alternative Derivation. We can give a more elegant and succinct, if less direct, derivation of the polynomial form of the scattering equations as follows. As we noted in section 1, the scattering equations (1.1) are equivalent to $p(z)^2$, as given by (1.15), vanishing everywhere as a function of z for $z_a, a\in A$, satisfying these equations, and this statement is equivalent to the vanishing of the polynomial of degree $N-2$,

$$F(z) = 2p(z)^2\prod_{a\in A}(z-z_a) = \sum_{\substack{S\subset A\\ |S|=2}} k^2_S\prod_{b\in\bar{S}}(z-z_b), \tag{2.16}$$

where $\bar{S} = \{a \in A : a \notin S\}$. Now,

$$F(z) = \sum_{m=0}^{N-2} z^{N-m-2} \sum_{\substack{U \subset A \\ |U|=m}} z_U \sum_{\substack{S \subset \bar{U} \\ |S|=2}} k_S^2. \tag{2.17}$$

From (2.15),

$$\sum_{\substack{S \subset \bar{U} \\ |S|=2}} k_S^2 = k_{\bar{U}}^2 = k_U^2 \tag{2.18}$$

so that the coefficients of z^{N-2} and z^{N-3} vanish and

$$F(z) = \sum_{m=2}^{N-2} z^{N-m-2} \tilde{h}_m. \tag{2.19}$$

establishing the equivalence of the polynomial form (1.2) to the original scattering equations (1.1).

Partially Fixing the Möbius Invariance. Because the set is equivalent to the equations $f_a = 0, a \in A$, the Möbius group acts on the solutions of the set of equations $\tilde{h}_m = 0, 2 \leq m \leq N-2$ (which define an algebraic variety in $\mathbb{C}^N$). As usual, it is convenient to fix this invariance, at least partially. We take $z_1 \to \infty, z_N \to 0$. This is facilitated by the fact that $\tilde{h}_m$ is linear in each of z_1, z_N separately. Write $A' = \{a \in A : a \neq 1, N\}$ and

$$h_m = \lim_{z_1 \to \infty} \frac{\tilde{h}_{m+1}}{z_1} = \sum_{\substack{S \subset A' \\ |S|=m}} k_{S_1}^2 z_S = \frac{1}{m!} \sum_{\substack{a_1, a_2, \ldots, a_m \in A' \\ a_i \text{ uneq.}}} \sigma_{a_1 a_2 \ldots a_m} z_{a_1} z_{a_2} \ldots z_{a_m}, \tag{2.20}$$

where

$$S_1 = S \cup \{1\}, \qquad \sigma_{a_1 a_2 \ldots a_m} = k_{1 a_1 a_2 \ldots a_m}^2. \tag{2.21}$$

h_m is an homogeneous polynomial of degree m in the variables $z_2, \ldots, z_{N-1}$, linear in each of them separately. Fixing $z_1 = \infty, z_N = 0$ breaks the Möbius group down to scalings $z_a \mapsto \lambda z_a, a \in A'$. At $z_1 = \infty, z_N = 0$, we can replace the equations $\tilde{h}_m = 0, 2 \leq m \leq N-2$ by $h_m = 0, 1 \leq m \leq N-3$. These equations define a (presumably) zero-dimensional projective variety in $\mathbb{CP}^{N-3}$, a set consisting typically of

$$\prod_{m=1}^{N-3} \deg h_m = (N-3)! \tag{2.22}$$

points by Bézout's Theorem.

Factorization. To see how the factorization, evident in (1.16) and (1.17), when $z_a \to z_0,\ a \in S$, implying $k_S^2 = 0$, manifests itself in terms of the polynomial equations, consider the equations $\tilde{h}_m = 0, 2 \leq m \leq N-2$, without specializing to $z_1 = \infty, z_N = 0$, and use the translation invariance that then holds to take $z_0 = 0$, without loss of generality. Let $n = |\bar{S}|$, where again $\bar{S} = \{a \in A : a \notin S\}$, and $z_a = \epsilon x_a + \mathcal{O}(\epsilon^2), a \in S$. Then

$$\tilde{h}_m = \tilde{h}_m^{\bar{S}0} + \mathcal{O}(\epsilon), \qquad 1 < m < n, \tag{2.23}$$

where

$$\tilde{h}_m^{\bar{S}0} = \sum_{\substack{U \subset \bar{S} \\ |U|=m}} k_U^2 z_U, \tag{2.24}$$

$\tilde{h}_n$ vanishes because $k_S^2 = 0$, and, for $n < m < N$,

$$\tilde{h}_m = \epsilon^{n-m} \tilde{h}_{m-n}^{S\infty} z_{\bar{S}} + \mathcal{O}(\epsilon^{m-n+1}), \tag{2.25}$$

where

$$\tilde{h}_r^{S\infty} = \frac{1}{r!} \sum_{\substack{U \subset S \\ |U|=r}} (k_{\bar{S}} + k_U)^2 x_U, \qquad 1 \le r < N - n. \tag{2.26}$$

$\tilde{h}_m^{\bar{S}0}$ is the appropriate form of $\tilde{h}_m$ for the polynomial scattering equations for $n+1$ particles with variables $z_a, a \in \bar{S}$ and 0, associated with momenta $k_a, a \in \bar{S}$, and k_S, while $\tilde{h}_m^{S\infty}$ is the appropriate form of h_m for the polynomial scattering equations for $N - n + 1$ particles with variables $z_b, b \in \ S$, and ∞, associated with momenta $k_b, b \in S$, and $k_{\bar{S}}$, thus demonstrating that the polynomial equations factorize as they should.

3 Solutions to the Scattering Equations

A Special Symmetric Configuration. For a given number of particles, N, if the dimension of space-time, $D \ge N - 1$, we choose momenta, k_a, so that the coefficients in h_m, for a given m, are all equal, by arranging that $k_a \cdot k_b = \mu, k_1 \cdot k_a = -\frac{1}{2}(N-3)\mu$, for $a, b \in A'$. [For example, in $\mathbb{R}^{1,N-2}$, take $k_1^0 = k_N^0 = \alpha, k_1^j = -k_N^j = \beta, k_a^0 = \gamma, k_a^j = \delta, j \neq a-1, k_a^{a-1} = -(N-3)\delta, 2 \le a \le N-1$, where $4\alpha^2 = 4(N-2)\beta^2 = (N-2)^2\gamma^2 = (N-3)(N-2)^3\delta^2$, so that $\mu = (N-2)^2\delta^2$. The configuration describes massless particles with momenta k_1, k_N annihilating in their center of mass frame to produce symmetrically $N-2$ massless particles with momenta $k_a, 2 \le a \le N-1$, the spatial components of whose momenta correspond to the vertices of a regular $(N-3)$-simplex.]

If its coefficients $k_{1a_1\ldots a_m}$ are all equal, h_m is proportional to the elementary symmetric function, $\Sigma_m^{A'}$, of (2.14). The conditions $\Sigma_m^{A'} = 0, 1 \le m \le N-3$, imply that

$$\prod_{b=2}^{N-1} (z - z_b) = z^{N-2} - \lambda^{N-2}, \qquad \text{where} \quad \lambda^{N-2} = \prod_{b=2}^{N-1} z_a. \tag{3.1}$$

Thus each $z = z_a$, $2 \le a \le N-1$, satisfies the equation $z^{N-2} = \lambda^{N-2}$ and, taking $\lambda = 1$, because only ratios of the z_a matter, the solutions of the scattering equations for these momenta are

$$z_a = \omega^{\rho(a-1)}, \qquad \rho \in \mathfrak{S}_{N-2}, \qquad 2 \le a \le N-1, \tag{3.2}$$

and ω is a complex $(N-2)$-th root of unity. The ratios z_a/z_2, $3 \le a \le N-1$, say, are thus given by the $N-3$ distinct complex $(N-2)$-th roots of unity, taken in some order, giving the $(N-3)!$ possible solutions. For this particular symmetric solution there is a symmetry under permuting the coordinates, which takes one solution into another, which is not present in general.

For further discussion of special configurations see [8], where Kalousios discusses kinematical configurations in which the solutions to the scattering equations can be identified with the zeros of Jacobi polynomials, and [3].

Solutions for $N = 4$ and $N = 5$. For $N = 4$, we have a simple linear equation determining z_3/z_2,

$$h_1 = \sigma_2 z_2 + \sigma_3 z_3 = 0, \quad z_3/z_2 = -\sigma_2/\sigma_3 = -k_1 \cdot k_2/k_1 \cdot k_3. \tag{3.3}$$

For $N = 5$, writing $(x, y, z) = (z_2, z_3, z_4)$, the equations

$$\begin{aligned} h_1 &= \sigma_2 x + \sigma_3 y + \sigma_4 z = 0 \\ h_2 &= \sigma_{23} xy + \sigma_{24} xz + \sigma_{34} yz = 0 \end{aligned} \tag{3.4}$$

yield a quadratic equation for z/y by elimination of x. This can be conveniently achieved by considering

$$h_{12}^{(x)} = \begin{vmatrix} h_1 & h_2 \\ h_1^x & h_2^x \end{vmatrix}, \tag{3.5}$$

where

$$h_m^x = \frac{\partial h_m}{\partial x}, \quad \text{and, more generally, write} \quad h_m^{a_1 \dots a_r} = \frac{\partial^r h_m}{\partial z_{a_1} \dots \partial z_{a_r}}. \tag{3.6}$$

Then, because h_m is linear in each z_i taken separately, $h_m^{a_1 \dots a_r} = 0$ if $a_i = a_j$ for any $i \neq j$. Additionally, $h_m^{a_1 \dots a_r} = 0$ if $r > m$. It follows immediately that $h_{12}^{(x)}$ is independent of x because

$$\frac{\partial h_{12}^{(x)}}{\partial x} = \begin{vmatrix} h_1^x & h_2^x \\ h_1^x & h_2^x \end{vmatrix} + \begin{vmatrix} h_1 & h_2 \\ h_1^{xx} & h_2^{xx} \end{vmatrix} = 0. \tag{3.7}$$

$h_{12}^{(x)}$ is quadratic in y and z (as can be seen, *e.g.*, by noting, using the approach of (3.7), that any third derivative of $h_{12}^{(x)}$ with respect to y and z vanishes) and $h_{12}^{(x)} = 0$ when $h_1 = h_2 = 0$, so that it provides the required quadratic equation for y/z. Given a solution to this equation, x/z is determined uniquely from $h_1 = 0$.

Solution for $N = 6$. This approach can be extended to $N = 6$ without much difficulty, where we expect a sextic equation to determine the ratios z_a/z_b. Write $(x, y, z, u) = (z_2, z_3, z_4, z_5)$ and

$$h_{123}^{(x|y)} = \begin{vmatrix} h_1 & h_2 & h_3 \\ h_1^x & h_2^x & h_3^x \\ h_1^{xy} & h_2^{xy} & h_3^{xy} \end{vmatrix}. \tag{3.8}$$

Then the technique of (3.7) shows that $\partial_x h_{123}^{(x|y)} = \partial_y^2 h_{123}^{(x|y)} = 0$, implying that $h_{123}^{(x|y)}$ is independent of x and linear in y; similarly it is cubic in z and u, suggesting that the required sextic might involve the product of two such expressions, $h_{123}^{(x|y)}h_{123}^{(y|x)}$. Indeed,

$$h_{123}^{(xy)} = \begin{vmatrix} h_1 & h_2 & h_3 \\ h_1^x & h_2^x & h_3^x \\ h_1^{xy} & h_2^{xy} & h_3^{xy} \end{vmatrix} \begin{vmatrix} h_1 & h_2 & h_3 \\ h_1^y & h_2^y & h_3^y \\ h_1^{xy} & h_2^{xy} & h_3^{xy} \end{vmatrix} - \begin{vmatrix} h_1 & h_2 & h_3 \\ h_1^x & h_2^x & h_3^x \\ h_1^y & h_2^y & h_3^y \end{vmatrix} \begin{vmatrix} h_1^x & h_2^x & h_3^x \\ h_1^y & h_2^y & h_3^y \\ h_1^{xy} & h_2^{xy} & h_3^{xy} \end{vmatrix}, \tag{3.9}$$

is indeed independent of both x and y because $\partial_x h_{123}^{(xy)} = \partial_y h_{123}^{(xy)} = 0$, and it vanishes when $h_1 = h_2 = h_3 = 0$; so $h_{123}^{(xy)} = 0$, provides the required sextic in z, u. Given one of the solutions for z/u to $h_{123}^{(xy)} = 0$, x/u and y/u are uniquely determined by $h_{123}^{(y|x)} = 0$ and $h_{123}^{(x|y)} = 0$, respectively.

The definition (3.9) can be rewritten as a single determinant,

$$h_{123}^{(xy)} = -\begin{vmatrix} h_1 & h_2 & h_3 & 0 & 0 & 0 \\ h_1^x & h_2^x & h_3^x & 0 & 0 & 0 \\ h_1^y & h_2^y & h_3^y & h_1 & h_2 & h_3 \\ h_1^{xy} & h_2^{xy} & h_3^{xy} & h_1^x & h_2^x & h_3^x \\ 0 & 0 & 0 & h_1^y & h_2^y & h_3^y \\ 0 & 0 & 0 & h_1^{xy} & h_2^{xy} & h_3^{xy} \end{vmatrix}, \tag{3.10}$$

which is a form that can be obtained from classical elimination theory [9].

4 Amplitudes in Terms of Polynomial Constraints

We want to rewrite (1.12) in terms of $h_m, 1 \le m \le N-3$, rather than $f_a, a \in A, a \ne i, j, k$. We can invert (2.1) to give

$$f_a(z,k) = \sum_{m=2}^{N-2} (Z^{-1})_{am} g_m(z,k), \tag{4.1}$$

where $Z_{ma} = z_a^{m+1}, a \in A, -1 \le m \le N-2$, noting that $g_{-1} = g_0 = g_1 = 0$. Since $Z^{-1} = (\det Z)^{-1}\mathrm{adj}Z$, the Jacobian of $\{f_a : a \in A, a \ne i, j, k\}$ with respect to $\{g_m : 2 \le m \le N-2\}$ is given by

$$\begin{aligned} \det\left(\frac{\partial f_a}{\partial g_m}\right)_{\substack{a\ne i,j,k \\ 2\le m\le N-2}} &= \frac{1}{(\det Z)^{N-3}} \det\left(\mathrm{adj}Z\right)_{\substack{a\ne i,j,k \\ 2\le m\le N-2}} = \frac{1}{\det Z}\det\left(Z\right)_{\substack{a=i,j,k \\ m=-1,0,1}} \\ &= (z_i - z_j)(z_i - z_k)(z_j - z_k)\prod_{a<b}(z_a - z_b)^{-1}, \end{aligned} \tag{4.2}$$

$$\det\left(\frac{\partial \tilde{h}_m}{\partial g_n}\right)_{2\le m,n\le N-2} = (-1)^{\frac{1}{2}N(N+1)} 2^{N-3}. \tag{4.3}$$

Thus, up to a sign and factors of 2,

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k) \prod_{m=2}^{N-2} \frac{1}{\tilde{h}_m(z,k)} \prod_{a<b} (z_a - z_b) \prod_{a\in A} \frac{dz_a}{(z_a - z_{a+1})^2} \Big/ d\omega. \tag{4.4}$$

Taking $z_1 \to \infty, z_2$ fixed, $z_N \to 0$,

$$\mathcal{A}_N = \oint_{\mathcal{O}} \Psi_N(z,k) \frac{z_2}{z_{N-1}} \prod_{m=1}^{N-3} \frac{1}{h_m(z,k)} \prod_{2\le a<b\le N-1} (z_a - z_b) \prod_{a=2}^{N-2} \frac{z_a dz_{a+1}}{(z_a - z_{a+1})^2}. \tag{4.5}$$

5 Möbius Transformations

Möbius Invariance of the Integrand for the Amplitude. It is straightforward to demonstrate directly the Möbius invariance of (4.4). The Möbius group is generated by translations, scaling and inversion, and, in this instance, demonstrating invariance under translation, $z_a \mapsto z_a + \epsilon$, involves a little more than the others. We have

$$\begin{aligned}
\tilde{h}_m(z-\epsilon) &= \sum_{\substack{S\subset A\\ |S|=m}} k_S^2 \prod_{a\in S} (z_a - \epsilon) && (5.1)\\
&= \sum_{\substack{S\subset A\\ |S|=m}} k_S^2 \sum_{r=0}^{m} (-\epsilon)^r \sum_{\substack{U\subset S\\ |U|=m-r}} z_U \\
&= \sum_{r=0}^{m} (-\epsilon)^r \sum_{\substack{U\subset A\\ |U|=m-r}} z_U \sum_{\substack{U\subset S\subset A\\ |S|=m}} k_S^2. && (5.2)
\end{aligned}$$

Given U with $|U| = n$,

$$\begin{aligned}
\sum_{\substack{U\subset V\subset A\\ |V|=n+1}} k_V^2 &= \sum_{b\in\bar{U}} (k_U + k_b)^2 \\
&= (N-n)k_U^2 + 2\sum_{b\in\bar{U}} k_U \cdot k_b \\
&= (N-n-2)k_U^2, \qquad \text{as } \; k_U = -k_{\bar{U}}. && (5.3)
\end{aligned}$$

Thus

$$\sum_{\substack{U\subset S\subset A\\ |S|=m}} k_S^2 = \frac{(N-m+r-2)!}{r!(N-m-2)!} k_U^2, \tag{5.4}$$

and so

$$\tilde{h}_m(z_1-\epsilon,\ldots,z_N-\epsilon) = \sum_{r=0}^{m-2} \frac{(N-m+r-2)!}{r!(N-m-2)!} (-\epsilon)^r\, \tilde{h}_{m-r}(z_1,\ldots,z_N), \tag{5.5}$$

as $\tilde{h}_0(z) = \tilde{h}_1(z) = 0$, from which it follows that (4.4) is unchanged under translations. Under scaling and inversion, respectively,

$$\tilde{h}_m(\lambda z_1, \ldots, \lambda z_N) = \lambda^m \tilde{h}_m(z_1, \ldots, z_N), \tag{5.6}$$

$$\tilde{h}_m(1/z_1, \ldots, 1/z_N) = \tilde{h}_{N-m}(z_1, \ldots, z_N)/z_A \tag{5.7}$$

and it may easily be checked that (4.4) is invariant under these as well, so that it is invariant under the full Möbius group.

Representations of the Möbius Algebra. Combining translation, T_ϵ, (5.5), with inversion, I, (5.7), gives the special conformal transformation, $S_\epsilon = IT_{-\epsilon}I$,

$$\tilde{h}_m\left(\frac{z_1}{1+\epsilon z_1}, \ldots, \frac{z_N}{1+\epsilon z_N}\right) = \sum_{r=0}^{N-m-2} \frac{(m+r-2)!}{r!(m-2)!}\, \epsilon^r\, \tilde{h}_{m+r}(z_1, \ldots, z_N) \prod_{a\in A} \frac{1}{1+\epsilon z_a}. \tag{5.8}$$

If L_{-1} denotes the generator of translations (5.5),

$$L_{-1} = -\sum_{a\in A} \frac{\partial}{\partial z_a}, \qquad L_{-1}\, \tilde{h}_m = -(N-m-1)\tilde{h}_{m-1}\,, \tag{5.9}$$

the generator of special conformal transformations (5.8),

$$L_1 = -\sum_{a\in A} z_a^2 \frac{\partial}{\partial z_a} + \Sigma_1^A, \qquad L_1\, \tilde{h}_m = (m-1)\tilde{h}_{m+1}. \tag{5.10}$$

With these definitions, the appropriate generator of scale transformations is

$$L_0 = -\sum_{a\in A} z_a \frac{\partial}{\partial z_a} + \frac{N}{2}, \qquad L_0\, \tilde{h}_m = (\tfrac{1}{2}N - m)\tilde{h}_m \tag{5.11}$$

so that

$$[L_1, L_{-1}] = 2L_0, \qquad [L_0, L_{\pm 1}] = \mp L_{\pm 1}. \tag{5.12}$$

The $\tilde{h}_m$, $2 \le m \le N-2$, form a basis for an $(N-3)$-dimensional representation of the Möbius algebra, *i.e.* a representation of 'Möbius spin' $\frac{1}{2}N - 2$. For this representation the quadratic Casimir,

$$L^2 \equiv L_0^2 - \tfrac{1}{2}L_1 L_{-1} - \tfrac{1}{2}L_{-1}L_1, \tag{5.13}$$

takes the value $(\frac{1}{2}N-2)(\frac{1}{2}N-1)$.

More generally, the ring, $\mathcal{R}^N$, of polynomials in $z_1, z_2, \ldots, z_N$, provides a graded infinite-dimensional representation space for (5.12). This is the N-fold tensor product of the representation

$$L_{-1} = -\frac{d}{dz}, \qquad L_0 = -z\frac{d}{dz} + \frac{1}{2}, \qquad L_1 = -z^2\frac{d}{dz} + z, \tag{5.14}$$

acting on the ring of single-variable polynomials, $\mathcal{R}^1$, in z. It is easy to see that the only invariant subspace of $\mathcal{R}^1$ for (5.14) is the two-dimensional subspace, $\mathcal{F}^1$, consisting of linear polynomials, *i.e.* that spanned by $\{1, z\}$, which carries a representation of 'Möbius spin' $\frac{1}{2}$, for which $L^2 = \frac{3}{4}$. [The quotient space $\mathcal{R}^1/\mathcal{F}^1$ provides an irreducible infinite-dimensional representation of the Möbius algebra with 'Möbius spin' $\frac{1}{2}$ and $L_0 \leq -\frac{3}{2}$.] Correspondingly, $\mathcal{R}^N$ has a Möbius invariant subspace $\mathcal{F}^N$, which is the tensor product of N copies of $\mathcal{F}^1$, one for each $z_a, a \in A$, and consists of polynomials that are linear in each of the z_a taken separately. $\mathcal{F}^N$ has dimension 2^N and has a basis consisting of the square-free monomials, $\{z_S : S \subset A\}$.

It is straightforward to show that, if the polynomial $\varphi(z) \in \mathcal{R}^N$ involves a monomial term with a factor z_a^n, with $n \geq 2$, then $(L_1)^M\varphi$ involves a monomial term with a factor z_a^{M+n}, $M \geq 0$, implying that φ can not be an element of a finite-dimensional Möbius invariant subspace of $\mathcal{R}^N$. Thus $\mathcal{F}^N$ is the largest finite-dimensional Möbius invariant subspace. It decomposes into irreducible subspaces of 'Möbius spin' $\frac{1}{2}N - n$, n an integer, $0 \leq n \leq \frac{1}{2}N$, of dimension $N - 2n + 1$, with the eigenvalues of L_0 being $\frac{1}{2}N - m$, where $n \leq m \leq N - n$.

Highest Weight Polynomials. Such irreducible subspaces, of 'Möbius spin' $\frac{1}{2}N - n$, are generated by the repeated action of L_1 from highest weight polynomials φ of degree n satisfying $L_{-1}\varphi = 0$, *i.e.* translation invariant polynomials in $\mathcal{F}^N$. The highest weight polynomial in an irreducible subspace is the one with the largest eigenvalue of L_0, or, equivalently, the lowest degree. If $\mathcal{F}_m^N$ denotes the subspace of $\mathcal{F}_N$ consisting of polynomials of degree m, so that $\mathcal{F}_m^N$ has a basis $\{z_S : S \subset A, |S| = m\}$, and $\mathcal{H}_n^N = \{\varphi \in \mathcal{F}_n^N : L_{-1}\varphi = 0\}$, the subspace of $\mathcal{F}_m^N$ consisting of highest weight polynomials,

$$\dim \mathcal{F}_m^N = \frac{N!}{m!(N-m)!}; \qquad \dim \mathcal{H}_n^N = \dim \mathcal{F}_n^N - \dim \mathcal{F}_{n-1}^N = \frac{N!(N-2n+1)}{n!(N-n+1)!}, \tag{5.15}$$

where $0 \leq m \leq N$, $0 \leq n \leq \frac{1}{2}N$. Thus,

$$\dim \mathcal{H}_0^N = 1; \qquad \dim \mathcal{H}_1^N = N - 1; \qquad \dim \mathcal{H}_2^N = \frac{N(N-3)}{2}. \tag{5.16}$$

For $\varphi \in \mathcal{F}_n^N$, write

$$\varphi = \sum_{\substack{S \subset A \\ |S|=n}} \varphi_S z_S, \qquad \varphi_S = \varphi_{i_1 \ldots i_n}, \quad \text{for} \quad S = \{i_1, \ldots, i_n\}. \tag{5.17}$$

[Here $\varphi_{i_1 \ldots i_n}$ is symmetric in $i_1, \ldots, i_n$ and vanishes if any two indices are equal.] Then $\varphi \in \mathcal{H}_n^N$, *i.e.* $L_{-1}\varphi = 0$, if and only if

$$\sum_{\substack{S \subset A, S \ni a \\ |S|=n}} \varphi_S = 0, \quad \text{for each} \quad a \in A, \quad \text{or, equivalently,} \quad \sum_{i_n \in A} \varphi_{i_1 \ldots i_n} = 0. \tag{5.18}$$

We can verify directly that the dimension of the space of tensors $\varphi_{i_1 \ldots i_n}$ satisfying these conditions is $\dim \mathcal{H}_n^N$ as given by (5.15). The action of L_1 on $\varphi \in \mathcal{H}_n^N$, defined by (5.17), is given by

$$L_1\varphi = \sum_{\substack{S\subset A\\|S|=n}} \varphi_S \sum_{a\in\bar{S}} z_a z_S = \sum_{\substack{U\subset A\\|U|=n+1}} \varphi_U z_U, \tag{5.19}$$

where, in general, we define

$$\varphi_U = \sum_{\substack{S\subset U\\|S|=n}} \varphi_S. \tag{5.20}$$

Then, if $m \geq n$, and

$$\varphi_m = \sum_{\substack{U\subset A\\|U|=m}} \varphi_U z_U, \tag{5.21}$$

we have

$$L_1\varphi_m = \sum_{\substack{U\subset A\\|U|=m}} \sum_{a\in\bar{U}} z_a z_U \sum_{\substack{S\subset U\\|S|=n}} .\varphi_S = (m-n+1)\varphi_{m+1} \tag{5.22}$$

So $L_1^r\varphi = r!\varphi_{n+r}$ and, using the algebra (5.12) and $L_{-1}\varphi = 0$, we can calculate $L_{-1}L_1^r\varphi$ and deduce that

$$L_1\varphi_m = (m-n+1)\varphi_{m+1}, \quad L_0\varphi_m = (\tfrac{1}{2}N-m)\varphi_m, \quad L_{-1}\varphi_m = -(N-m-n+1)\varphi_{m-1}, \tag{5.23}$$

where $\varphi_{n-1} = 0$, and, as we see below, $\varphi_{N-n+1} = 0$, and

$$L^2\varphi_m = (\tfrac{1}{2}N-n)(\tfrac{1}{2}N-n+1)\varphi_m, \tag{5.24}$$

in agreement with the 'Möbius spin' of the multiplet with highest weight polynomial $\varphi \equiv \varphi_n$ being $\frac{1}{2}N - n$, and with (5.9)–(5.11) for $n = 2$. It follows from (5.18) that

$$\begin{aligned}\varphi_U &= \frac{1}{n!}\sum_{a_1,\dots,a_n\in U} \varphi_{a_1\dots a_n} = -\frac{1}{n!}\sum_{\substack{a_1,\dots,a_{n-1}\in U\\a_n\in\bar{U}}} \varphi_{a_1\dots a_n}\\ &= \frac{(-1)^n}{n!}\sum_{a_1,\dots,a_n\in\bar{U}} \varphi_{a_1\dots a_n} = (-1)^n\varphi_{\bar{U}}.\end{aligned} \tag{5.25}$$

It follows that $I\varphi_m = (-1)^n\varphi_{N-m}$, from which it follows that $\varphi_m = 0$ if $m > N-n$, consistent with the fact that the $N-2n+1$ polynomials $\varphi_n, \varphi_{n+1}, \dots, \varphi_{N-n}$ are a basis for the invariant subspace generated from the highest weight polynomial $\varphi \equiv \varphi_n$.

A basis for $\mathcal{F}_0^N$ is provided by $\varphi \equiv \varphi_0 = 1$, and then $\varphi_m = \Sigma_m^A$, $0 \leq m \leq N$, the elementary symmetric polynomial defined by (2.14). An element of $\mathcal{F}_1^N$ is of the form $\varphi \equiv \varphi_1 = \sum_{a\in A}\lambda_a z_a$, where $\sum_{a\in A}\lambda_a = 0$, and then

$$\varphi_m = \sum_{a\in A}\lambda_a z_a \Sigma_{m-1}^{A_a}, \qquad \text{where } \ A_a = \{b\in A : b\neq a\}. \tag{5.26}$$

Uniqueness. If $\varphi \in \mathcal{H}_n^N$, the invariant linear subspace generated from φ has dimension $N-2n+1$, and so it is the case $n = 2$ that is of particular interest to us because it is only in this case that the conditions $\varphi_m = 0$ provide the right number of constraints to determine a finite number of points in $\mathbb{CP}^{N-1}$, up to Möbius transformations. Further, from (5.15), $\dim \mathcal{H}_2^N = \frac{1}{2}N(N-3)$, which, if the dimension of space-time is sufficiently high, is the number of independent degrees of freedom, $k_a \cdot k_b$, of light-like momenta $k_a, a \in A$, satisfying momentum conservation (up to Lorentz transformation), with φ_U determined by

$$\varphi_U = k_U^2 \tag{5.27}$$

as in (2.15). This implies that a Möbius invariant set of equations in $\mathbb{CP}^{N-1}$ determining a finite set of points, up to Möbius transformation, has to have the form of the polynomial scattering equations (1.2), at least if the Möbius transformations are realized as in (5.9)–(5.11).

To see more directly that, if $\varphi \in \mathcal{H}_2^N$, then φ_m is of the form (1.2), take vectors, $k_a, a \in A$, summing to zero, with $\{k_1, \ldots, k_{N-1}\}$ linearly independent, define a scalar product by

$$\langle k_a, k_b \rangle = \langle k_b, k_a \rangle = \lambda_{ab}, \quad 1 \le a < b \le N-1; \quad \langle k_a, k_a \rangle = 0, \quad 1 \le a \le N-1. \tag{5.28}$$

Then, taking $\varphi_{a,b} = (k_a + k_b)^2$, φ_U, as defined by (5.20), is given by (5.27) and $\varphi_m = \tilde{h}_m$, defined as in (2.11).

6 Massive Particles

In the discussion of massless particles, the order of the particles is not relevant. To discuss massive particles [4], we need to select a cyclic ordering on A, which we take to be the ordering $1, 2, \ldots, N, 1$. We replace the definition of f_a in (1.1) by

$$f_a(z,k) = \sum_{\substack{b \in A \\ b \ne a}} \frac{k_a \cdot k_b}{z_a - z_b} + \frac{\mu^2}{2(z_a - z_{a-1})} + \frac{\mu^2}{2(z_a - z_{a+1})}, \quad a \in A. \tag{6.1}$$

Using (6.1) in (2.1) to define g_m, it is again the case that Möbius invariance of this system of equations implies that g_{-1}, g_0, g_1 all vanish identically, provided that $k_a^2 = \mu^2$ for each $a \in A$.

$$\begin{aligned} g_m(z,k) &= \frac{1}{2} \sum_{\substack{a,b \in A \\ a \ne b}} k_a \cdot k_b \frac{z_a^{m+1} - z_b^{m+1}}{z_a - z_b} + \frac{\mu^2}{2} \sum_{a \in A} \frac{z_{a+1}^{m+1} - z_a^{m+1}}{z_{a+1} - z_a} \\ &= \frac{1}{2} \sum_{\substack{a,b \in A \\ a \ne b}} k_a \cdot k_b \sum_{r=0}^{m} z_a^r z_b^{m-r} + \frac{\mu^2}{2} \sum_{a \in A} \sum_{r=0}^{m} z_{a+1}^r z_a^{m-r}. \end{aligned} \tag{6.2}$$

Then, following section 2, we obtain, as in (2.5),

$$g_2(z,k) = \frac{1}{2} \sum_{\substack{a,b \in A \\ a < b}} (k_{ab}^2 - n_{ab}\mu^2) z_a z_b, \tag{6.3}$$

where $n_{a,a\pm1}=1,\ n_{ab}=2,\ b\neq a\pm1$, provided that $k_a^2=\mu^2,\ a\in A$, and, as in (2.6),

$$g_3(z,k)=\frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}}(k_{ab}^2-n_{ab}\mu^2)z_a^2z_b; \tag{6.4}$$

$$\sum_{\substack{c,a\in A\\ c\neq a}}z_cz_a^3f_a(z,k)=g_2(z,k)\sum_{c\in A}z_c-g_3(z,k)=\frac{1}{12}\sum_{\substack{a,b,c\in A\\ a,b,c\ \text{uneq.}}}(k_{abc}^2-n_{abc}\mu^2)z_az_bz_c; \tag{6.5}$$

$$\begin{aligned}g_2(z,k)\sum_{c\in A}z_c&=\frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}}(k_{ab}^2-n_{ab}\mu^2)z_a^2z_b+\frac{1}{4}\sum_{\substack{a,b,c\in A\\ a,b,c\ \text{uneq.}}}(k_{ab}^2-n_{ab}\mu^2)z_az_bz_c\\&=\frac{1}{2}\sum_{\substack{a,b\in A\\ a\neq b}}(k_{ab}^2-n_{ab}\mu^2)z_a^2z_b+\frac{1}{12}\sum_{\substack{a,b,c\in A\\ a,b,c\ \text{uneq.}}}(k_{abc}^2-n_{abc}\mu^2)z_az_bz_c,\end{aligned} \tag{6.6}$$

where $n_{abc}=n_{ab}+n_{ac}+n_{bc}-3$, provided that $k_b^2=\mu^2$ for $b\in A$. [$n_{abc}^2=3$, if none of a,b,c are adjacent, $n_{abc}=2$, if just two them are, and $n_{abc}=1$, if a,b,c are sequential (in some order).] Thus, as in (2.8),

$$\sum_{\substack{c,a\in A\\ c\neq a}}z_cz_a^3f_a(z,k)=g_2(z,k)\sum_{c\in A}z_c-g_3(z,k)=\frac{1}{12}\sum_{\substack{a,b,c\in A\\ a,b,c\ \text{uneq.}}}(k_{abc}^2-n_{abc}\mu^2)z_az_bz_c. \tag{6.7}$$

For $S\subset A$, let n_S denote the number of disjoint strings of adjacent elements of S with the given cyclic ordering on A. Then $1\leq n_S\leq|S|$, with $n_S=1$ if and only if the elements of S are sequential. Then, as in (2.10),

$$\sum_{\substack{a_0,a_2,\dots,a_m\in A\\ a_i\ \text{uneq.}}}z_{a_2}\dots z_{a_m}z_{a_0}^2f_{a_0}=-\frac{(m-1)!}{2}\sum_{\substack{S\subset A\\ |S|=m}}(k_S^2-n_S\mu^2)z_S. \tag{6.8}$$

We can deduce (6.3)–(6.8) directly from section 2 by introducing a 'fictitious' auxiliary space, orthogonal to space-time, and space-like vectors $\kappa_a, a\in A$, with $\kappa_a^2=-1,\kappa_a\cdot\kappa_{a+1}=\frac{1}{2},\kappa_a\cdot\kappa_b=0,|a-b|>1,a,b\in A$. [The κ_a, $a\in A$, span an $(N-1)$-dimensional space.] Write $\tilde{k}_a=k_a+\mu\kappa_a$; then

$$\tilde{k}_a^2=k_a^2-\mu^2,\quad a\in A,\qquad\qquad \tilde{k}_S^2=k_S^2-n_S\mu^2,\quad S\subset A. \tag{6.9}$$

Thus, the massless condition $\tilde{k}^2=0$ is equivalent to the mass-shell condition $k^2=\mu^2$, and substituting $\tilde{k}_a$ for k_a in (1.1) yields (6.1) and similarly (6.3), (6.4) and (6.8) follow from (2.5), (2.6) and (2.10), respectively.

The equations $f_a=0, a\in A$, are equivalent to the equations $\tilde{h}_m=0, 2\leq m\leq N-2$, where

$$\tilde{h}_m=\sum_{\substack{S\subset A\\ |S|=m}}(k_S^2-n_S\mu^2)z_S. \tag{6.10}$$

The $\tilde{h}_m$ are related to the g_m, defined by (2.1) with f_a given by (6.1), by (2.13). The Möbius invariance of this system can be demonstrated directly as in section 4.

As in section 2, it is convenient to fix the Möbius invariance partially by letting $z_1 \to \infty$ and $z_N \to 0$. Again, writing $A' = \{a \in A : a \neq 1, N\}$ and $S_1 = S \cup \{1\}$,

$$h_m = \lim_{z_1 \to \infty} \frac{\tilde{h}_{m+1}}{z_1} = \sum_{\substack{S \subset A' \\ |S|=m}} (k_{S_1}^2 - n_S \mu^2) z_S, \tag{6.11}$$

we can replace the equations $\tilde{h}_m = 0, 2 \leq m \leq N-2$ by $h_m = 0, 1 \leq m \leq N-3$.

7 Four-Dimensional Space-Time

Fairlie and Roberts [5], who were concerned with four-dimensional space-time, noted that a particular solution of the scattering equations (1.1) is given by

$$z_a = \frac{k_a^3 - k_a^0}{k_a^2 + ik_a^1}, \qquad 1 \leq a \leq N, \tag{7.1}$$

for $D = 4$ and for all N. The demonstration of this can conveniently be expressed using twistor variables, $\pi_a, \bar{\pi}_a$, defined by

$$k_a^\mu \sigma_\mu = \begin{bmatrix} k_a^0 - k_a^3 & -ik_a^1 + k_a^2 \\ -ik_a^1 - k_a^2 & k_a^0 + k_a^3 \end{bmatrix} = \pi_a \bar{\pi}_a^T, \qquad \pi_a = \begin{bmatrix} \pi_a^1 \\ \pi_a^2 \end{bmatrix}, \qquad \bar{\pi}_a = \begin{bmatrix} \bar{\pi}_a^1 \\ \bar{\pi}_a^2 \end{bmatrix}. \tag{7.2}$$

In terms of these,

$$z_a = \pi_a^1/\pi_a^2, \qquad k_a \cdot k_b = \langle \pi_a, \pi_b \rangle [\bar{\pi}_a, \bar{\pi}_b], \tag{7.3}$$

with

$$\langle \pi_a, \pi_b \rangle = \pi_a^1 \pi_b^2 - \pi_a^2 \pi_b^1, \quad [\bar{\pi}_a, \bar{\pi}_b] = \bar{\pi}_a^1 \bar{\pi}_b^2 - \bar{\pi}_a^2 \bar{\pi}_b^1, \quad z_a - z_b = \frac{\langle \pi_a, \pi_b \rangle}{\pi_a^2 \pi_b^2}. \tag{7.4}$$

Then

$$\sum_{b \neq a} \frac{k_a \cdot k_b}{z_a - z_b} = \sum_b \pi_a^2 \pi_b^2 [\bar{\pi}_a, \bar{\pi}_b] = 0 \tag{7.5}$$

by momentum conservation,

$$\sigma_\mu \sum_b k_b^\mu = \sum_b \bar{\pi}_b \bar{\pi}_b^T = 0. \tag{7.6}$$

Another solution to the scattering equations in $D = 4$, not equivalent under Möbius transformations to that of [5] (except for $N = 4$), is given by $z_a = \bar{\pi}_a^1/\bar{\pi}_a^2$. These are two rational explicit solutions which exist for all N. As noted in [10], these solutions are can be associated with the MHV and anti-MHV amplitudes, and the others can be associated other combinations of helicities. [The scattering equations in four-dimensional space have also been discussed recently by Weinzierl [11].] Although CHY introduced the scattering equations in order to describe tree amplitudes for Yang-Mills and gravity [2, 10], perhaps more basically, they also describe scalar particles [3, 4]. It seems that the other solutions can be associated with the division of the particles into n, say, positive helicities and $N-m$ negative helicities. However, perhaps strangely, the solutions only depend on how many positive and negative helicities there are, *i.e.* they depend on n, not on which particles are assigned positive helicity and which negative.

To see this, as in [12] (see also [13]), we divide the N indices $a \in A$ into m 'positive' indices $i \in \mathcal{P}$ and n 'negative' indices $r \in \mathcal{N}$. Then the link variables c_{ir}, introduced by Arkani-Hamed, Cachazo, Cheung and Kaplan [14, 15], satisfy

$$\pi_i = \sum_{s\in\mathcal{N}} c_{is}\pi_s, \qquad \bar\pi_r = -\sum_{j\in\mathcal{P}} \bar\pi_j c_{jr}, \qquad i \in \mathcal{P},\ r \in \mathcal{N}, \tag{7.7}$$

and these are related to the parameters of twistor string theory [16] – [18] by

$$c_{ir} = \frac{\lambda_i}{\lambda_r(z_i - z_r)}, \qquad i \in \mathcal{P},\ r \in \mathcal{N}. \tag{7.8}$$

Then, for $i \in \mathcal{P}$,

$$\sum_a \frac{k_i\cdot k_a}{z_i - z_a} = \sum_r \frac{\langle\pi_i,\pi_r\rangle[\bar\pi_i,\bar\pi_r]}{z_i - z_r} + \sum_j \frac{\langle\pi_i,\pi_j\rangle[\bar\pi_i,\bar\pi_j]}{z_i - z_j} \tag{7.9}$$

$$\begin{aligned}
\sum_r \frac{\langle\pi_i,\pi_r\rangle[\bar\pi_i,\bar\pi_r]}{z_i - z_r} &= -\sum_{rsj} \frac{c_{is}\langle\pi_s,\pi_r\rangle[\bar\pi_i,\bar\pi_j]c_{jr}}{z_i - z_r} = \sum_{rsj} \frac{\lambda_i\lambda_j\langle\pi_r,\pi_s\rangle[\bar\pi_i,\bar\pi_j]}{\lambda_r\lambda_s(z_i-z_r)(z_i-z_s)(z_j-z_r)}\\
&= \tfrac{1}{2}\sum_{rsj} \frac{\lambda_i\lambda_j\langle\pi_r,\pi_s\rangle[\bar\pi_i,\bar\pi_j](z_r-z_s)}{\lambda_r\lambda_s(z_i-z_r)(z_i-z_s)(z_j-z_r)(z_j-z_s)}\\
\sum_j \frac{\langle\pi_i,\pi_j\rangle[\bar\pi_i,\bar\pi_j]}{z_i - z_j} &= \sum_{rsj} c_{ir}c_{js}\frac{\langle\pi_r,\pi_s\rangle[\bar\pi_i,\bar\pi_j]}{z_i - z_j} = \sum_{rsj} \frac{\lambda_i\lambda_j\langle\pi_r,\pi_s\rangle[\bar\pi_i,\bar\pi_j]}{\lambda_r\lambda_s(z_i-z_j)(z_i-z_r)(z_j-z_s)}\\
&= -\tfrac{1}{2}\sum_{rsj} \frac{\lambda_i\lambda_j\langle\pi_r,\pi_s\rangle[\bar\pi_i,\bar\pi_j](z_r-z_s)}{\lambda_r\lambda_s(z_i-z_r)(z_j-z_s)(z_i-z_s)(z_j-z_r)},
\end{aligned} \tag{7.10}$$

where it is understood that r, s are summed over $\mathcal{N}$ and j over $\mathcal{P}$, with $j \neq i$. Thus the scattering equations (1.1) hold for $a = i \in \mathcal{P}$ and a similar argument shows that it holds if $a \in \mathcal{N}$.

Spradlin and Volovich [13] conjectured that the number of solutions of the equations (7.7) and (7.8) is given by the Eulerian number $\left\langle {N-3 \atop m-2} \right\rangle$, $2 \leq m \leq N-2$, and CHY have given a demonstration of this [10]. These numbers sum to $(N-3)!$, the number of solutions of (1.1). This implies that the solutions depend on an assignment of a number of positive and negative helicities, but must be independent of which particular momenta are associated with positive helicities and which negative, otherwise there would be too many solutions. To demonstrate this independence, consider moving i into $\mathcal{N}$ and r into $\mathcal{P}$,

$$\begin{aligned}\pi_r = \frac{1}{c_{ir}}\pi_i - \sum_{s\neq r}\frac{c_{is}}{c_{ir}}\pi_s &= \frac{\lambda_r}{\lambda_i}(z_i - z_r)\pi_i - \sum_{s\neq r}\frac{\lambda_r(z_i - z_r)}{\lambda_s(z_i - z_s)}\pi_s \\ &= \frac{\mu_r}{\mu_i(z_r - z_i)}\pi_i + \sum_{s\neq r}\frac{\mu_r}{\mu_s(z_r - z_s)}\pi_s, \end{aligned} \tag{7.11}$$

$$\begin{aligned}\pi_j = \frac{c_{jr}}{c_{ir}}\pi_i + \sum_{s\neq r}\left[c_{js} - \frac{c_{jr}c_{is}}{c_{ir}}\right]\pi_s &= \frac{\lambda_j(z_i - z_r)}{\lambda_i(z_j - z_r)}\pi_i + \sum_{s\neq r}\frac{\lambda_j}{\lambda_s}\frac{(z_r - z_s)(z_j - z_i)}{(z_j - z_s)(z_i - z_s)(z_j - z_r)}\pi_s \\ &= \frac{\mu_j}{\mu_i(z_j - z_i)}\pi_i + \sum_{s\neq r}\frac{\mu_j}{\mu_s(z_j - z_s)}\pi_s \end{aligned} \tag{7.12}$$

where

$$\lambda_r = \frac{\mu_r}{z_r - z_i}, \quad \lambda_i = \mu_i(z_i - z_r), \quad \lambda_s = \mu_s\frac{z_s - z_r}{z_s - z_i}, \quad \lambda_j = \mu_j\frac{z_j - z_r}{z_j - z_i} \tag{7.13}$$

So, defining $\tilde{\mathcal{P}} = \{r; j \in \mathcal{P}, j \neq i\}$ and $\tilde{\mathcal{N}} = \{i; s \in \mathcal{N}, s \neq r\}$, we have

$$\pi_\ell = \sum_{t\in\tilde{\mathcal{N}}}\tilde{c}_{\ell t}\pi_t, \quad \ell \in \tilde{\mathcal{P}}, \qquad \tilde{c}_{\ell t} = \frac{\mu_\ell}{\mu_t(z_\ell - z_t)}; \tag{7.14}$$

and, similarly, we can show

$$\bar{\pi}_t = \sum_{\ell\in\tilde{\mathcal{P}}}\bar{\pi}_\ell\tilde{c}_{\ell t}, \quad t \in \tilde{\mathcal{N}}. \tag{7.15}$$

Thus, remarkably a solution (z_a) to the scattering equations obtained from a division of the N indices $a \in A$ into n 'positive' indices $i \in \mathcal{P}$ and $N - n$ 'negative' indices $r \in \mathcal{N}$, does not depend on the choice of $\mathcal{P}, \mathcal{N} \subset A$, but only on n [as explicitly seen above in the $n = 2$ and $n = N - 2$ 'MHV' and 'anti-MHV' cases].

8 Comments

A general understanding of the polynomial form of the scattering equations was provided by the discussion of Möbius invariance in section 5. If the Möbius algebra acts as in (5.9)–(5.11), the polynomial scattering equations are the only sets of homogeneous polynomials, related by Möbius transformations, determining a finite set of points.

In general, for a given highest weight polynomial φ, satisfying $L_{-1}\varphi = 0, L_0\varphi = (\frac{1}{2}N - n)\varphi$, the equations, $\varphi_m = 0, n \leq m \leq N - n$, defined as in (5.21), determine an algebraic variety in the complex projective space $\mathbb{CP}^{N-1}$ on which the Möbius group acts. [We shall assume $\varphi_S \neq 0, S \subset A, n \leq |S| \leq N - n$.] For $n = 0$, where, up to a constant, $\varphi = 1$, this is empty. For $1 \leq n \leq \frac{1}{2}N$, we can use the Möbius invariance to send $z_1 \to \infty, z_N \to 0$, as in (2.20) and (6.11),

$$h_m^\varphi = \lim_{z_1\to\infty} \frac{1}{z_1}\varphi_{m+1} = \sum_{\substack{S\subset A' \\ |S|=m}} \varphi_{S_1} z_S, \qquad n-1 \le m \le N-n-1, \tag{8.1}$$

where, again, $S_1 = S \cup \{1\}$ and $A' = \{a \in A : a \neq 1, N\}$. The homogeneous equations $h_m^\varphi = 0, n-1 \le m \le N-n-1$, define a complex projective variety, $\mathcal{V}_\varphi$, in $\mathbb{CP}^{N-3}$, which we expect typically to have dimension $2n-4$. For $n=1$, h_0^φ is constant and there are no solutions, $\mathcal{V}_\varphi = \varnothing$. As we have discussed, $n=2$ provides precisely the scattering equations, for all possible light-like momenta satisfying momentum conservation, with typically $\mathcal{V}_\varphi$ containing $(N-3)!$ points. It would be interesting to determine conditions on φ so that, for $n=2$, $\mathcal{V}_\varphi$ contains precisely this number of distinct points, and, for $n>2$, $\dim \mathcal{V}_\varphi = 2n-4$. It would also be interesting to know whether the varieties $\mathcal{V}_\varphi$, for $n>2$, have any physical significance.

We may realize differently the Möbius group by changing the action of inversion from (5.7) to

$$\phi(z_1, \dots, z_N) \mapsto (z_A)^K \phi(1/z_1, \dots, 1/z_N), \tag{8.2}$$

where K is a positive integer, and then, in place of (5.9)–(5.11), which correspond to $K=1$, we have

$$L_n = \sum_{a\in A} \left[-z_a^{n+1}\frac{\partial}{\partial z_a} + K\frac{n+1}{2} z_a^n \right] \tag{8.3}$$

defining the action of translation, scaling and the special conformal transformation for $n = -1, 0, 1$, respectively. In this case, for $N=1$, a single variable, the finite-dimensional invariant subspace has 'Möbius spin' $\frac{1}{2}K$. For general N it will be the tensor product of N copies of this space and so will have dimension $(K+1)^N$; it consists of all polynomials involving no power of z_a higher than z_a^K for each $a \in A$. Clearly, we may consider Möbius invariant systems of polynomial equations in this more general context but we have not yet analyzed these.

The CHY expressions for the tree amplitudes in scalar, gauge and gravity theories involve, in effect, summing a rational function (their integrand) over the finite projective variety $\mathcal{V}_\varphi$, where φ is the degree 2 highest weight polynomial whose coefficients are the scalar products of the external (light-like) momenta. Although the points of $\mathcal{V}_\varphi$ are, in general, irrational, the result is a rational function of the coefficients of φ and the rational integrand. It would be desirable to have formula for this result, and an understanding of it, in terms of the algebro-geometric properties of $\mathcal{V}_\varphi$, perhaps along the lines of elimination theory [19]. The understanding of scattering equations in terms of (1.15) and (2.16) may provide insight into possible extensions of the scattering equations from the Riemann sphere to the torus.

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